A Characterization of Expansiveness of Shift Homeomorphisms of Inverse Limits of Graphs(General Topology and related fields)

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1. Introduction.

All spaces under consideration are assumed to be metric. By a *continuum*, we mean a compact connected nondegenerate space. Let X be a compact metric space with metric d. A homeomorphism $f: X \to X$ is called *expansive* if there is a positive number $c > 0$ (called an *expansive constant for* f) such that if x and y are different points of X, then there is an integer $n = n(x,y) \in \mathbb{Z}$ such that

$$d(f^n(x), f^n(y)) > c.$$

Expansiveness does not depend on the choice of metric d of X. In [15], Mañé proved that if $f: X \to X$ is an expansive homeomorphism of a compact metric space X, then $\dim X < \infty$ and every minimal set is 0-dimensional. This result shows that there is some restriction on which spaces admit expansive homeomorphisms. we are interested in the following problem [3]: What kinds of continua admit expansive homeomorphisms? In [19], Williams first showed that there is a 1-dimensional continuum admiting an expansive homeomorphisms. In fact, he proved that the shift homeomorphism of the dyadic solenoid is expansive. From continuum theory in topology, we know that inverse limit spaces yield powerful techniques for constructing complicated spaces and maps from simple spaces and maps. Naturally, we are also interested in the next problem: What kinds of maps induce expansiveness of the shift homeomorphisms? It is well-known that positively expansive maps induce expansiveness of the shift homeomorphisms(e.g.,see [19]). In [6], Jacobson and Utz asserted that shift homeomorphisms of the inverse limit of every surjective map on an arc is not expansive (see [1,p. 648] for the complete proof).

It is known that "Plykin's attractors" are 1-dimensional continua in the plane R^2 and are examples of Williams' 1-dimensional expanding attractors, homeomorphisms on which are not only expansive homeomorphisms but even hyperbolic diffeomorphisms (see [20] and [21]). Also, Plykin's attractors can be represented as inverse limits of maps $g: K \to K$ of graphs such that the shift homeomorphisms of the maps are expansive (see [20, p. 243] and [21, p. 121]). In [10], we proved that if an onto map $f: G \to G$ of a graph G is null-homotopic, then the shift homeomorphism $\tilde{f}$ of f is not expansive. In particular, shift homeomorphisms of tree-like continua are not expansive. Also, we proved that for any graph G containing a simple closed curve, there is an onto map $f: G \to G$ such that the shift homeomorphism $\tilde{f}$ of f is expansive. Hence, there is a G-like continuum X admiting an expansive homeomorphism.

In this note, we investigate expansive homeomorphisms from a point of view of inverse limits.

2. A characterization of expansiveness of shift homeomorphisms of inverse limits of graphs.

Let X be a compact metric space with metric d. By *the hyperspace of* X, we mean $C(X) = \{A \mid A$ is a nonempty subcontinuum of $X\}$ with the *Hausdorff metric* d_H, i.e., $d_H(A,B) = \inf\{\varepsilon > 0 \mid U_\varepsilon(A) \supset B$ and $U_\varepsilon(B) \supset A\}$, where $U_\varepsilon(A)$ denotes the ε-neighborhood of A in X. It is well-known that if X is a continuum, then C(X) is arcwise connected.

Let f be an expansive homeomorphism of a compact metric space X with an expansive constant $c > 0$. If $\varepsilon > 0$, let W^s_ε and W^u_ε be *the local stable* and *unstable* families of subcontinua of X defined by

$$W^s_\varepsilon = \{A \in C(X) \mid \operatorname{diam} f^n(A) \leq \varepsilon \text{ for any } n \geq 0\},$$
$$W^u_\varepsilon = \{A \in C(X) \mid \operatorname{diam} f^{-n}(A) \leq \varepsilon \text{ for any } n \geq 0\}.$$

Also, define families W^s and W^u of *stable* and *unstable subcontinua* as

$$W^s = \{A \in C(X) \mid \lim_{n\to\infty} \operatorname{diam} f^n(A) = 0\},$$
$$W^u = \{A \in C(X) \mid \lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0\}.$$

Then we know that if $\varepsilon \le c$, then $W^s = \cup\{f^{-n}(A) \mid A \in W^s_\varepsilon,\ n \ge 0\}$ and $W^u = \cup\{f^n(A) \mid A \in W^u_\varepsilon,\ n \ge 0\}$ (see [15, p.315]).

Let X be a compact metric space with metric d. For a map $f\colon X \to X$, let

$$(X,f) = \{(x_i)_{i=1}^{\infty} \mid x_i \in X,\ f(x_{i+1}) = x_i\ (i \ge 1)\}.$$

Define a metric $\tilde{d}$ for (X,f) by

$$\tilde{d}(\tilde{x},\tilde{y}) = \Sigma_{i=1}^{\infty} d(x_i,y_i)/2^i,\text{ where } \tilde{x} = (x_i)_{i=1}^{\infty},\ \tilde{y} = (y_i)_{i=1}^{\infty} \in$$
(X,f).

Then the space (X,f) is called *the inverse limit of the map* $f\colon X \to X$. Define a map $\tilde{f}$ by

$$\tilde{f}((x_i)_{i=1}^{\infty}) = (f(x_i))_{i=1}^{\infty} = ((x_{i-1})_{i=1}^{\infty}.$$

Then $\tilde{f}$ is a homeomorphism and called *the shift homeomorphism of the map* f.

Let $p_n\colon(X,f) \to X_n = X$ be the projection defined by $p_n((x_i)_{i=1}^{\infty}) = x_n$.

Let A be a closed subset of a compact metric space X with metric d. A map $f\colon X \to X$ is *positively expansive on* A if there is a positive number $c > 0$ such that if $x, y \in A$ and $x \ne y$, then there is a natural number $n \geq 0$ such that

$$d(f^n(x), f^n(y)) > c.$$

Such a positive number c is called *a positively expansive constant for* $f|A$. If $f: X \to X$ is a positively expansive map on X, then f is called *positively expansive.* Clearly, if $f: X \to X$ is a positively expansive map on A, then $f|A: A \to X$ is locally injective.

Let $\mathcal{A}$ be a finite closed covering of a compact metric space X. A map $f: X \to X$ is called *a positively pseudo-expansive map with respect to* $\mathcal{A}$ if

(P_1) f is positivey expansive on A for each $A \in \mathcal{A}$, and

(P_2) for the case $A, B \in \mathcal{A}$ and $A \cap B \neq \phi$, one of the following two conditions holds:

(*) f is positively expansive on $A \cup B$.

(**) If f is not positively expansive on $A \cup B$, then there is a natural number $k \geq 1$ such that for any $A', A'' \in \mathcal{A}$ with $A' \cap A'' \neq \phi$,

$$f^k(A' \cup A'') \cap (A - B) = \phi \quad \text{or} \quad f^k(A' \cup A'') \cap (B - A) = \phi.$$

A map $f: X \to X$ is called *positively pseudo-expansive* if f is positively pseudo-expansive with respect to some finite closed covering $\mathscr{A}$ of X. This notion is important for constructing various kinds of expansive homeomorphisms(e.g., see (2.1), (2.2) and (2.3)). By the definitions, positively expansive maps imply positively pseudo-expansive maps, but the converse assertion is not true. Concerning positively pseudo-expansive maps of graphs (= 1-dimensional compact connected polyhedra), we know the following facts (see [10]).

(2.1). *If* $f: X \to X$ *be a positively pseudo-expansive map of a compact metric space* X, *then the shift homeomorphism* $\tilde{f}$ *of* f is *expansive.*

(2.2). *Let* G *be a graph. Then* G *admit a positively pseudo-expansive map if and only if* G *contains a simple closed curve.*

(2.3). *Let* $f: G \to G$ *be an onto map of a graph* G. *If* f *is null-homotopic, then the shift homeomorphism* $\tilde{f}$ *is not expansive, hence* f *is not a positively pseudo-expansive map.*

(2.4). *If* $f: G \to G$ *is a positively expansive map of a graph* G, *then the inverse limit space* (G,f) *of* f *can not be embedded in the plane, but there are various kinds of positively pseudo-expansive maps*

such that the inverse limits of the maps can be embeddable in the plane.

Now, we shall give the following characterization of expansiveness of shift homeomorphisms of inverse limits of graphs as follows [12].

(2.5) Theorem. *Let* $f: G \to G$ *be an onto map of a graph* G. *Then the shift homeomorphism* $\tilde{f}$ *of* f *is expansive if and only if* f *is a positively pseudo-expansive map with respect to some finite closed over* $\mathcal{A}$ *of* G, *where* $\mathcal{A} = \{e \mid e$ *is an edge of some simplical complex* K^* *such that*
$|K^*| = G\}$.

(2.6) Corollary. *Let* $f: G \to G$ *be an onto map of a graph* G.
If the shift homeomorphism $\tilde{f}$ *of* f *is expansive, then there is a positive number* $\alpha > 0$ *such that if* $A \in C((G,f))$ *and* diam $A \le \alpha$, $A \in W^u = \{D \in C((G,f)) \mid \lim_{n\to\infty} \text{diam } \tilde{f}^{-1} D) = 0\}$. *Also, for any* $x \in (G,f)$, *there is an arc* A *in* (G,f) *containing* x *such that* $A \in W^u$.

(2.7) Corollary. *Let* $f: X \to X$ *be an onto map of a graph* G.
If $\tilde{f}$ *is an expansive homeomorphism, then* $W^s = \{\{x\} \mid x \in (G,f)\}$, *where* $W^s = \{A \in C((G,f)) \mid \lim_{n\to\infty} \text{diam } \tilde{f}^n(A) = 0\}$.

(2.8) Remark. In the statement of (2.5), the cases of n-dimensional polyhedra($n \ge 2$) are not true. It is well-known there is an expansive homeomorphism f of the 2-torus $T = S^1 \times S^1$. The shift homeomorphism $\tilde{f}$ of f is topologically conjugate to f.

Note that if $f: X \to X$ is an expansive homeomorphism of a Peano continuum X, then for any open set U, $f|U$ is not positively expansive. Then $\tilde{f}$ is expansive, but f is not positively pseudo-expansive.

(2.9) Remark. There is an expansive homeomorphism F of a 1-dimensional continuum X such that F and F^{-1} can not be represented by shift homeomorphisms of maps of graphs.
Let $S^1 = \{z \in \mathbf{C} |\ |z| = 1\}$, where $\mathbf{C}$ is the set of complex numbers and let $f: S^1 \to S^1$ be a map defined by $f(e^{i\theta}) = e^{i2\theta}$. Then $\tilde{f}$ is an expansive homeomorphism of the dyadic solenoid (S^1,f). Let p be the fixed point of $\tilde{f}$. Let $D = (S^1,f)$ and let $X = (D_1,p_1) \vee (D_2,p_2)$ be the one point union of two copies of (D,p).
Define a map $F: X \to X$ by $F(x) = \tilde{f}(x)$ for $x \in D_1$ and $F(x) = \tilde{f}^{-1}(x)$ for $x \in D_2$. By (2.7), F is a desired expansive homeomorphism.

(2.10) Example. In [20] and [21], it was shown that for each $n = 3,4,5,\ldots$, there exist a graph G_n and an onto map $g_n: G_n \to G_n$ such that $\pi_1(G_n) = \overbrace{\mathbf{Z}*\mathbf{Z}*\ldots*\mathbf{Z}}^{n}$, g_n is a homotopy equivalence, the shift homeomorphism $\tilde{g}_n$ of g_n is expansive (hence, g_n is a positively pseudo-expansive map) and $(G_n,g_n) \subset \mathbf{R}^2$.
Here, we give an example which implies that the case $n = 2$ is not true. Let G_2 be the one point union of two

oriented circles $\vec{a}$ and $\vec{b}$. Note that $\pi_1(G_2) = \mathbb{Z}*\mathbb{Z}$. Define a map $g_2\colon G_2 \to G_2$ by

$$\vec{a} \longrightarrow \vec{a}*\vec{b}*\vec{a},$$
$$\vec{b} \longrightarrow \vec{b}*\vec{a}.$$

Then we can easily see that g_2 is a positively pseudo-expansive and g_2 is a homotopy equivalence. In fact, the homotopy inverse $h\colon G_2 \to G_2$ is defined by

$$\vec{a} \longrightarrow \vec{a}*(\vec{b})^{-1},$$
$$\vec{b} \longrightarrow \vec{b}*\vec{b}*(\vec{a})^{-1}.$$

Hence (G_2, g_2) is movable.

Also, we can easily see that the case $n = 1$ is not true, i.e., for any graph G_1 with $\pi_1(G_1) = \mathbb{Z}$, there are no positively pseudo-expansive maps which are homotopy equivalences. The case $n = 0$ is not true. In fact, there are no positively pseudo-expansive maps on trees [11]. But, we can prove that if $f\colon G \to G$ is a positively pseudo-expansive map on G with $\pi_1(G) = \mathbb{Z}*\mathbb{Z}$, then the inverse limit (G,f) of f can not be embedded in the plane, which implies that Plykin's example is best possible concerning expansiveness of shift homeomorphisms and embedding into the plane.

3. Stable and unstable subcontinua of expansive homeomorphisms.
In [9], we proved that there are no expansive homeomorphisms on Suslinian continua. In this section, we give more precise result which is related to the stable and unstable properties [12].

(3.1) Theorem. *Let* X *be a compact metric space with* dim X $\geq$ 1. *If* f: X $\to$ X *is an expansive homeomorphism, then there is a closed subset* Z *of* X *such that each component of* Z *is nondegenerate, the space of components of* Z *is a Cantor set, the decomposition of* Z *into components is continuous* (i.e., *upper-semi and lower-semi continuous*), *and all components of* Z *are contained in* W^s *or* W^u.

3. Problems.
The following problems remain open.

Problem 1. For each n = 1, 2, 3, does there exist a plane continuum X_n admiting an expansive homeomorphism such that $\mathbf{R}^2 - X_n$ has n-components? Also, is there a tree-like continuum admiting expansive homeomorphisms? (Note that there are no Peano continua in the plane which admit expansive homeomorphisms and there are no hereditarily decomposable tree-like (or circle-like) continua admiting expansive homeomorphisms.)

Problem 2. Does there exist a 1-dimensional Peano continuum admiting an expansive homeomorphism? How about Menger's universal curve?

Problem 3. If a continuum X admits an expansive homeomorphism, does X contain an indecomposable (nondegenerate) subcontinuum? (Note that if a continuum X is tree-like or circle-like, this problem has a positive answer.)

References

[1] N. Aoki, Topological dynamics, Topics in general topology (K. Morita and J. Nagata, Eds.), Elsevier Science Publishers B. V. (1989), 625-740.

[2] B. F. Bryant, Unstable self-homeomorphisms of a compact space, Vanderbilt University, Thesis, 1954.

[3] W. Gottschalk, Minimal sets; an introduction to topological dynamics, Bull. Amer. Math. Soc. 64 (1958), 336-351.

[4] W. Gottschalk and G. Hedlund, Topological dynamics, Amer. Math. Soc. Colloq. 34 (1955).

[5] K. Hiraide, Expansive homeomorphisms of compact surfaces are pseudo Anosov, Osaka J. Math. to appear.

[6] J. F. Jacobson and W. R. Utz, The nonexistence of expansive homeomorphisms of a closed 2-cell, Pacific J. Math. 10 (1960), 1319-1321.

[7] H. Kato, The nonexistence of expansive homeomorphisms of 1-dimensional compact ANRs, Proc. Amer. Math. Soc. to appear.

[8] _______, The nonexistence of expansive homeomorphisms of Peano continua in the plane, Topology and its appl. to appear.

[9] _______, The nonexistence of expansive homeomorphisms of Suslinian continua, J. Math. Soc. Japan, to appear.

[10] _______, On expansiveness of shift homeomorphisms of inverse limits of graphs, Fund. Math. to appear.

[11] _______, The nonexistence of expansive homeomorphisms of dendroids, Fund. Math. to appear.

[12] _______, On expansive homeomorphisms in continuum theory, preprint.

[13] H. Kato and K. Kawamura, A class of continua which admit no expansive homeomorphisms, Rocky Mountain J. Math. to appear.

[14] K. Kawamura, A direct proof that Peano continuum with a free arc admits no expansive homeomorphisms, Tsukuba J. Math. 12 (1988), 521-524.

[15] R. Mañé, Expansive homeomorphisms and topological dimension, Trans. Amer. Math. Soc. 252 (1979), 313-319.

[16] T. O'Brien and W. Reddy, Each compact orientable surface of positive genus admits an expansive homeomorphism, Pacific J. Math. 35 (1970), 737-741.

[17] W. Reddy, The existence of expansive homeomorphisms of manifolds, Duke Math. J. 32 (1965), 627-632.

[18] ________, Expansive canonical coordinates are hyperbolic, Topology and its appl. 15 (1983), 205-210.

[19] R. F. Williams, A note on unstable homeomorphisms, Proc. Amer. Math. Soc. 6 (1955), 308-309.

[20] R. V. Plykin, Sources and Sinks of A-Diffeomorphisms of Surfaces, Math. USSR. Sbornik 23 (1974), 233-253.

[21] _______, On the geometry of hyperbolic attractors of smooth cascades, Russian Math. Surveys 39 (1984), 85-131.